Reflection Sequences

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Let $X = (x_0, x_1, \dots, x_{n-1})$ be a circular sequence of real numbers and suppose their sum $S = \sum_{i=0}^{n-1} x_i$ is strictly positive. If, for some i, $x_i < 0$, then there is a *legal reflection* R_i for X defined as the operation which transforms X into the circular sequence $X' = (x_0', x_1', \dots, x_{n-1}')$ obtained from X by replacing x_i by $-x_i$ and by subtracting $|x_i|$ from the two neighbors of x_i in X. That is, $x_i' = -x_i$, $x_{i-1}' = x_{i-1} + x_i$, $x_{i+1}' = x_{i+1} + x_i$ and $x_j' = x_j$ for all $j, 0 \leqslant j < n$, $j \neq i-1, i, i+1$, where all indices are reduced modulo n. Notice that $\sum_{i=0}^{n-1} x_i' = \sum_{i=0}^{n-1} x_i$.

A *reflection sequence* for X is a finite or infinite sequence $X = X_0, X_1, \dots$ of circular sequences and a sequence $R^1, \dots$ of reflections, such that R^i is a legal reflection for X_{i-1} that transforms it into X_i, $1 \leqslant i$. If all the elements of X_t are nonnegative, that is, if there is no legal reflection for X_t, then we say that X_t is *stable*.

If, for example, $n = 5$ and $X = (1, -2, -3, 8, 5)$, then one can easily check that $X = X_0$, $X_1 = (-1, 2, -5, 8, 5)$, $X_2 = (-1, -3, 5, 3, 5)$, $X_3 = (-4, 3, 2, 3, 5)$, $X_4 = (4, -1, 2, 3, 1)$ and $X_5 = (3, 1, 1, 3, 1)$ with the corresponding reflections R_1, R_2, R_1, R_0, R_1 a reflection sequence for X which terminates in the stable configuration X_5.

A celebrated problem in the 1986 International Mathematical Olympiad (cf. [**1**]) asserts that for $n = 5$ and for any circular sequence of integers $X = (x_0, x_1, x_2, x_3, x_4)$ (whose sum is positive) any reflection sequence for X must be finite. Equivalently, if we start from X and apply, repeatedly, legal reflections, we eventually reach a stable circular sequence.

This result holds for general n as stated in the following Proposition.

PROPOSITION 1. *If $X = (x_0, \dots, x_{n-1})$ is a circular sequence of $n (\geqslant 2)$ integers whose sum $S = \sum_{i=0}^{n-1} x_i$ is positive, then any reflection sequence for X is finite.*

To prove Proposition 1 we need some notation. A sequence of consecutive numbers modulo n is called an *arc*. We denote by $\langle i, k \rangle$ the arc $\langle i, k \rangle = (i, i+1, \dots, k-1, k)$ where all numbers are reduced modulo n. The *complementary arc* of $\langle i, k \rangle$ is $\langle k+1, i-1 \rangle$. For a sequence $X = (x_0, \dots, x_{n-1})$ and an arc $\langle i, k \rangle$ we define the arc-sum of $\langle i, k \rangle$ (with respect to X) by $S_X(\langle i, k \rangle) = \sum_{j \in \langle i, k \rangle} x_j$. Clearly, the sum of the arc-sum of any arc with the arc-sum of its complementary arc is precisely $\sum_{j=0}^{n-1} x_j = S$.

Suppose, now, that $X' = (x_0', \dots, x_{n-1}')$ is obtained from X by a legal reflection R_i. Clearly the arc-sum of any arc whose intersection with $\langle i-1, i+1 \rangle$ is of cardinality 0 or 3 does not change. The arc-sums of the two complementary arcs $\langle i, i \rangle$ and $\langle i+1, i-1 \rangle$ change from x_i and $S - x_i$ to $-x_i$ and $S + x_i$, respectively. The arc-sums of other arcs that intersect $\langle i-1, i+1 \rangle$ in 1 or 2 elements only interchange, that is, $S_{X'}(\langle k, i \rangle) = S_X(\langle k, i-1 \rangle)$, $S_{X'}(\langle i, k \rangle) = S_X(\langle i+1, k \rangle)$, $S_{X'}(\langle k, i-1 \rangle) = S_X(\langle k, i \rangle)$ and $S_{X'}(\langle i+1, k \rangle) = S_X(\langle i, k \rangle)$.

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Define $f(X) = \Sigma |S_X(\langle i, k\rangle)|^2$ where the summation ranges over all arcs $\langle i, k\rangle$. By the preceding paragraph

$$f(X') - f(X) = (S + x_i)^2 + x_i^2 - (S - x_i)^2 - x_i^2 = 4Sx_i < 0.$$

Therefore, if $X = X_0, X_1, X_2, \dots$ is a reflection sequence for X, then $(f(X_i))_{i \geqslant 1}$ is a strictly decreasing sequence of nonegative integers and thus must be finite. This completes the proof of Proposition 1. □

The conclusion of Proposition 1 clearly holds even if we replace the assumption that X is a sequence of integers by the assumption that it is a sequence of rationals. (Simply write each x_i as an integral multiple of some rational q and apply the result for integers.) The proof, however, does not imply finiteness for general real sequences. We next show that finiteness, in fact, holds for every real circular sequence X. Moreover, any two reflection sequences for X that terminate in stable configurations have the same length.

PROPOSITION 2. *Let $X = (x_0, \dots, x_{n-1})$ be a circular sequence of $n(\geqslant 2)$ real numbers whose sum $S = \Sigma_{i=0}^{n-1} x_i$ is positive. Then any reflection sequence for X is finite. Moreover, the length of any reflection sequence for X that terminates in a stable configuration is independent of the sequence (and can be easily computed from the x_i's).*

Proof. Associate each pair of complementary arcs $\lambda = \{\langle i, k\rangle, \langle k+1, i-1\rangle\}$ with a pair of real numbers $\{a_\lambda, b_\lambda\}$, where $a_\lambda = S_X(\langle i, k\rangle)$ and $b_\lambda = S_X(\langle k+1, i-1\rangle)$. Clearly $a_\lambda + b_\lambda = S$. If, for some λ, either a_λ or b_λ is negative, say, $a_\lambda < 0$, we define a *legal switch* Q_λ as the operation that transforms $\{a_\lambda, b_\lambda\}$ into $\{-a_\lambda, b_\lambda + 2a_\lambda\}$ and does not change all the other pairs. A set of pairs $\{\{a_\lambda, b_\lambda\}\}$ is *stable* if $a_\lambda, b_\lambda \geqslant 0$ for all λ, that is, if there is no legal switch for it. Obviously the operators Q_λ commute. The key idea is that any reflection sequence corresponds to a sequence of legal switches on the pairs $\{a_\lambda, b_\lambda\}$. Indeed, by the arguments given in the proof of Proposition 1, any legal reflection R_i corresponds to a legal switch on the pair $\lambda = \{\langle i, i\rangle, \langle i+1, i-1\rangle\}$, together with a permutation on the other pairs. Therefore, the only effect of the legal reflection R_i on the multiset of pairs $\{a_\lambda, b_\lambda\}$, where λ ranges over all pairs of complementary arcs, is obtained by applying the legal switch Q_λ for $\lambda = \{\langle i, i\rangle, \langle i+1, i-1\rangle\}$.

Returning to the commuting operators Q_λ it is easy to check that if $a_\lambda < 0$ and one can apply k repeated legal switches Q_λ to $\{a_\lambda, b_\lambda\}$, then

$$Q_\lambda^k\{(a_\lambda, b_\lambda)\} = \begin{cases} \{kb_\lambda + (k+1)a_\lambda, -(k-1)b_\lambda - ka_\lambda\} & k \text{ even} \\ \{-ka_\lambda - (k-1)b_\lambda, kb_\lambda + (k+1)a_\lambda\} & k \text{ odd.} \end{cases}$$

It follows that the number of consecutive legal switches that can be applied to a pair $\{a_\lambda, b_\lambda\}$ with $a_\lambda < 0$ is precisely

$$\min\{k \geqslant 1 : kb_\lambda + (k+1)a_\lambda \geqslant 0\} = \lceil |a_\lambda|/(b_\lambda + a_\lambda)\rceil = \lceil |a_\lambda|/S\rceil.$$

As the operators Q_λ commute this implies that the length of any sequence of legal switches that terminates in a stable configuration is precisely

$$\sum_{a_\lambda < 0} \left\lceil \frac{|a_\lambda|}{S} \right\rceil + \sum_{b_\lambda < 0} \left\lceil \frac{|b_\lambda|}{S} \right\rceil. \tag{1}$$

Since a circular sequence is stable if and only if all its arc-sums are nonnegative (that is, iff a_λ, $b_\lambda \geqslant 0$ for every pair of arcs λ), the length of any reflection sequence for X that terminates in a stable configuration is independent of the sequence and is given in (1). □

In the example we gave we considered the circular sequence $X = (1, -2, -3, 8, 5)$. One can easily check that for this sequence $S = 9$ and the set of all negative arc-sums is $\{-2, -3, -5, -1, -4\}$. By (1), the length of any reflection sequence for X that terminates in a stable configuration is $\lceil 2/9 \rceil + \lceil 3/9 \rceil + \lceil 5/9 \rceil + \lceil 1/9 \rceil + \lceil 4/9 \rceil = 5$. One can easily find several distinct such sequences. Somewhat surprisingly, they all end in the same stable configuration $(3, 1, 1, 3, 1)$. We next show that this is always the case.

PROPOSITION 3. *Let $X = (x_0, \ldots, x_{n-1})$ be a circular sequence of n real numbers whose sum is positive. Then any reflection sequence $\mathscr{R}$ for X that terminates in a stable configuration, terminates in the same configuration. Moreover, the set of indices $\{i: R_i$ occurs at least once in $\mathscr{R}\}$ is independent of $\mathscr{R}$, but the exact number of times each legal reflection R_i appears in the sequence $\mathscr{R}$ may depend on the specific sequence.*

Proof. One can easily check that the operators R_j satisfy the following relations:

(2) For nonadjacent i, k $R_i R_k = R_k R_i$, and
(3) For adjacent i, k $R_i R_k R_i = R_k R_i R_k$.

Suppose the first half of Proposition 3 is false and there is a circular sequence X and two reflection sequences for X which terminate in two distinct stable configurations. Among all such counterexamples X to the first half of Proposition 3, choose one, say, $\overline{X}$, such that the length t of some (and hence every) reflection sequence that brings it to a stable configuration is minimum. Let $R^1 = R^{1,1}, R^{1,2}, \ldots, R^{1,t}$ and $R^2 = R^{2,1}, R^{2,2}, \ldots, R^{2,t}$ be two reflection sequences for $\overline{X}$ which terminate in two distinct stable configurations Y and Z, respectively. By the minimality of t in the choice of $\overline{X}$, $R^{1,1} \neq R^{2,1}$. (Indeed, otherwise $R^{1,1}\overline{X} = R^{2,1}\overline{X}$ is another counterexample with two reflection sequences of length $t - 1$ each that terminate in the two distinct stable configurations Y and Z, contradicting the minimality of t.) Furthermore, by the minimality of t, all the terminating reflection sequences starting from $R^{1,1}\overline{X}$ end in the same stable configuration, which is Y. In particular, any terminating sequence starting from $R^{2,1}R^{1,1}\overline{X}$ ends in Y. Similarly, any terminating sequence starting from $R^{1,1}R^{2,1}\overline{X}$ ends in Z. If $R^{2,1}$ and $R^{1,1}$ are not adjacent we conclude, from (2), that $Y = Z$, a contradiction. For adjacent $R^{1,1}$ and $R^{2,1}$ we argue similarly, using (3) instead of (2) to obtain the same contradiction.

This proves the first half of Proposition 3. The second half is proved analogously. We omit the details. □

Remark. A different approach to the problem considered in this note, and certain generalizations of it, appears in **[2]**, where reflection processes are analyzed using the theory of Kac-Moody algebras and their buildings.

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Counting the Rationals

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Cantor's proof of the countability of the positive rationals has great appeal. One sees the idea literally at a glance. On the other hand the construction counts all ordered pairs of positive integers so that each positive rational is counted infinitely many times, and if one wants, say, the 10^{15}th positive rational in Cantor's list, one has to keep counting for a considerable time.

Here we offer a direct way of counting the positive rationals. Given m/n, we can assume that m and n are relatively prime. Let $m = p_1^{e_1} \cdots p_k^{e_k}$, $n = q_1^{f_1} \cdots q_l^{f_l}$, be the prime-number decompositions of m and n. The counting function is defined by: $f(1) = 1$ and

$$f\left(\frac{m}{n}\right) = p_1^{2e_1} \cdots p_k^{2e_k} q_1^{2f_1-1} \cdots q_l^{2f_l-1}.$$

f is clearly 1-1 and onto. The 10^{15}th positive rational in this list is 10^{-8}.